SCHRIFTENREIHE DER FAKULTÄT FÜR MATHEMATIK

Spectral Analysis for Differential Operators of Variable Orders on Star-type Graphs: General Case.

by
Vjacheslav Anatoljevich Yurko

SM-UDE-774 2014

Eingegangen am 06.01.2014

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V. Yurko

Abstract. We study inverse spectral problems for ordinary differential equations on compact star-type graphs when differential equations have different orders on diferent edges. As the main spectral characteristics we introduce and study the so-called Weyl-type matrices which are generalizations of the Weyl function (m-function) for the classical Sturm-Liouville operator. We provide a procedure for constructing the solution of the inverse problem and prove its uniqueness.

Key words: geometrical graphs, differential operators, inverse spectral problems

AMS Classification: 34A55 34L05 47E05

1. Introduction. We study inverse spectral problems for ordinary differential equations of variable orders on compact star-type graphs. More precisely, differential equations have different orders on diferent edges. Boundary value problems on graphs (spatial networks, trees) often appear in natural sciences and engineering (see [1]-[6]). Inverse spectral problems consist in recovering operators from their spectral characteristics. We pay attention to the most important nonlinear inverse problems of recovering coefficients of differential equations (potentials) provided that the structure of the graph is known a priori.

Differential equations of variable orders on graphs arise in various problems in mathematics as well as in applications (see, for example, [7] and the references therein). In particular, we mention transverse oscillation problems for such structures as cable-stayed bridges, masts with cable supports and others.

For *second-order* differential operators on compact graphs inverse spectral problems have been studied fairly completely in [8]-[14] and other works. Inverse problems for *higher-order* differential operators on graphs were investigated in [15]-[16]. We note that inverse spectral problems for second-order and for higher-order ordinary differential operators on *an interval* (finite or infinite) have been studied by many authors (see the monographs [17]-[23] and the references therein).

In [24]-[25] the inverse spectral problem is considered for a very particular case of variable order differential equations on star-type graphs, when there are only two differential equations with different orders. In this paper we study the general case of variable order differential equations on star-type graphs. More precisely, all edges are divided into m groups in each of which differential equations have different orders. Moreover, we consider general matching conditions in the interior vertex. This general case produces new qualitative difficulties related to the formulation and the solution of the inverse problem.

As the main spectral characteristics in this paper we introduce and study the so-called Weyl-type matrices which are generalizations of the Weyl function (m-function) for the classical Sturm-Liouville operator (see [26]), of the Weyl matrix for higher-order differential operators on an interval introduced in [22]-[23], and generalizations of the Weyl-type matrices for higher-order differential operators on graphs (see [15]-[16]). We show that the specification of the Weyl-type matrices uniquely determines the coefficients of the differential equation on the graph, and we provide a constructive procedure for the solution of the inverse problem from the given Weyl-type matrices. For studying this inverse problem we develope the ideas of the method of spectral mappings [22]-[23]. The obtained results are natural generalizations of the well-known results on inverse problems for differential operators on an interval and on graphs, and in particular, generalizations of the results from [24]-[25].

2. Formulation of the inverse problem. Consider a compact star-type graph T in $\mathbf{R}^\omega$ with the set of vertices $V=\{v_0,\ldots,v_p\}$ and the set of edges $\mathcal{E}=\{e_1,\ldots,e_p\}$, where $v_1,\ldots,v_p$ are the boundary vertices, v_0 is the internal vertex, and $e_j=[v_j,v_0]$, $j=\overline{1,p}$, $\bigcap\limits_{j=1}^{p} e_j=\{v_0\}$. Let l_j be the length of the edge e_j. Each edge $e_j\in\mathcal{E}$ is parameterized by the parameter $x_j\in[0,l_j]$ such that $x_j=0$ corresponds to the boundary vertices $v_1,\ldots,v_p$, and $x_j=l_j$ corresponds to the internal vertex v_0. An integrable function Y on T may be represented as $Y=\{y_j\}_{j=\overline{1,p}}$, where the function $y_j(x_j)$ is defined on the edge e_j.

Fix $m=\overline{1,p}$. Let n_i,p_i, $i=\overline{1,m}$, be positive integers such that

$$n_1>n_2>\ldots>n_m>1,\quad 0<p_1<p_2<\ldots<p_{m-1}<p_m:=p,$$

and put $n_{m+1}:=1$, $p_0:=0$. Consider the differential equations on T:

$$y_j^{(n_i)}(x_j)+\sum_{\mu=0}^{n_i-2}q_{\mu j}(x_j)y_j^{(\mu)}(x_j)=\lambda y_j(x_j),\quad x_j\in(0,l_j),\ i=\overline{1,m},\ j=\overline{p_{i-1}+1,p_i},\tag{1}$$

where λ is the spectral parameter, $q_{\mu j}(x_j)$ are complex-valued integrable functions. Thus, the differential equations have order n_i on the edges e_j, $j=\overline{p_{i-1}+1,p_i}$. We call $q_j=\{q_{\mu j}\}$ the potential on the edge e_j, and we call $q=\{q_j\}_{j=\overline{1,p}}$ the potential on the graph T. Denote $\mu_i=p_i-p_{i-1}$, $i=\overline{1,m}$. Clearly, $\mu_1+\ldots+\mu_m=p$, and μ_i is the number of edges where differential equations have order n_i.

Fix $i=\overline{1,m}$, $j=\overline{p_{i-1}+1,p_i}$. Let $\{C_{kj}(x_j,\lambda)\}$, $k=\overline{1,n_i}$, be the fundamental system of solutions of equation (1) on the edge e_j under the initial conditions $C_{kj}^{(\mu-1)}(0,\lambda)=\delta_{k\mu}$, $k,\mu=\overline{1,n_i}$. Here and in the sequel, $\delta_{k\mu}$ is the Kronecker symbol. For each fixed $x_j\in[0,l_j]$, the functions $C_{kj}^{(\mu-1)}(x_j,\lambda)$, $k,\mu=\overline{1,n_i}$, are entire in λ of order $1/n_i$. Consider the linear forms

$$U_{j\nu}(y_j)=\sum_{\mu=0}^{\nu}\gamma_{j\nu\mu}y_j^{(\mu)}(l_j),\quad j=\overline{1,p},$$

where $\gamma_{j\nu\mu}$ are complex numbers, $\gamma_{j\nu}:=\gamma_{j\nu\nu}\neq 0$, $\nu=\overline{0,n_i-1}$ for $j=\overline{p_{i-1}+1,p_i}$. The linear forms $U_{j\nu}$ will be used in matching conditions at the internal vertex v_0 for boundary value problems and for the correspondung special solutions of equation (1).

Denote $\langle n\rangle:=(|n|+n)/2$, i.e. $\langle n\rangle=n$ for $n\geq 0$, and $\langle n\rangle=0$ for $n\leq 0$. Fix $i=\overline{1,m}$; $s=\overline{p_{i-1}+1,p_i}$; $\xi=\overline{i,m}$; $k=\overline{n_{\xi+1},n_\xi-1}$; $\mu=\overline{k,n_i}$. Let $L_{sk\mu}$ be the boundary value problem for equation (1) on the graph T with the boundary conditions

$$y_s(0)=\ldots=y_s^{(k-2)}(0)=y_s^{(\mu-1)}(0)=0,\tag{2}$$

$$y_j^{(r)}(0)=0,\quad r=\overline{0,\langle n_l-k-1\rangle};\ l=\overline{1,m},\ j=\overline{p_{l-1}+1,p_l},\ j\neq s,\tag{3}$$

and the matching conditions

$$U_{p_l,\nu}(y_{p_l})=U_{j\nu}(y_j),\ l=\overline{\xi,m},\ j=\overline{1,p_l-1},\ \nu=\overline{n_{l+1}-1,\min(k-1,n_l-2)},\tag{4}$$

$$\sum_{j=1}^{p_\xi}U_{j\nu}(y_j)=0,\ \nu=\overline{k,n_\xi-1};\quad \sum_{j=1}^{p_l}U_{j\nu}(y_j)=0,\ l=\xi-1,\ldots,i,\ \nu=\overline{n_{l+1},n_l-1}.\tag{5}$$

Note that the number of conditions in (2)-(5) is $n_1\mu_1+\cdots+n_m\mu_m$. Matching conditions (4)-(5) are generalizations of classical matching conditions for Sturm-Liouville operators on graphs [9], matching conditions for higher-order differential operators on graphs [15], and matching conditions for variable order differential operators on graphs [24].

Fix $i = \overline{1,m}$, $s = \overline{p_{i-1}+1, p_i}$, $k = \overline{1, n_i - 1}$. We introduce the solutions $\Psi_{sk} = \{\psi_{skj}\}_{j=\overline{1,p}}$ of equation (1) on the graph T as follows. Let $\xi = \overline{i,m}$; $k = \overline{n_{\xi+1}, n_\xi - 1}$. Then Ψ_{sk} satisfies the boundary conditions

$$\psi_{sks}^{(\eta-1)}(0) = \delta_{k\eta}, \quad \eta = \overline{1,k}, \tag{6}$$

$$\psi_{skj}^{(r)}(0) = 0, \quad r = \overline{0, \langle n_l - k - 1\rangle};\ l = \overline{1,m},\ j = \overline{p_{l-1}+1, p_l},\ j \neq s, \tag{7}$$

and the matching conditions at the vertex v_0 :

$$U_{p_l,\nu}(\psi_{sk,p_l}) = U_{j\nu}(\psi_{skj}),\ l = \overline{\xi, m},\ j = \overline{1, p_l - 1},\ \nu = \overline{n_{l+1} - 1, \min(k-1, n_l - 2)}, \tag{8}$$

$$\sum_{j=1}^{p_\xi} U_{j\nu}(\psi_{skj}) = 0,\ \nu = \overline{k, n_\xi - 1};\quad \sum_{j=1}^{p_l} U_{j\nu}(\psi_{skj}) = 0,\ l = \xi - 1, \dots, i,\ \nu = \overline{n_{l+1}, n_l - 1}. \tag{9}$$

The function Ψ_{sk} is called the Weyl-type solution of order k with respect to the boundary vertex v_s . Define additionally $\psi_{sn_is}(x_s, \lambda) := C_{n_is}(x_s, \lambda)$.

We introduce the matrices $M_s(\lambda)$, $s = \overline{p_{i-1}+1, p_i}$, $i = \overline{1,m}$, as follows:

$$M_s(\lambda) = [M_{sk\mu}(\lambda)]_{k,\mu=\overline{1,n_i}}, \quad M_{sk\mu}(\lambda) := \psi_{sks}^{(\mu-1)}(0, \lambda).$$

It follows from the definition of Ψ_{sk} that $M_{sk\mu}(\lambda) = \delta_{k\mu}$ for $k \geq \mu$, and $\det M_s(\lambda) \equiv 1$. The matrix $M_s(\lambda)$ is called the Weyl-type matrix with respect to the boundary vertex v_s . The inverse problem is formulated as follows. Fix $N = \overline{1,m}$.

Inverse problem 1. Given $\{M_s(\lambda)\}$, $s = \overline{1,p} \setminus p_N$, construct q on T.

We note that the notion of the Weyl-type matrices M_s is a generalization of the notion of the Weyl function (m-function) for the classical Sturm-Liouville operator ([20], [26]) and is a generalization of the notion of Weyl matrices introduced in [15], [16], [22]-[24] for higher-order differential operators on an interval and on graphs. Thus, Inverse Problem 1 is a generalization of the well-known inverse problems for differential operators on an interval and on the graphs.

We also note that in Inverse problem 1 we do not need to specify all matrices $M_s(\lambda)$, $s = \overline{1,p}$; one of them can be omitted. This last fact was first noticed in [9], where the inverse problem was solved for the Sturm-Liouville operators on an arbitrary tree.

Section 3 provides an example of the notions introduced above. In section 4 properties of the Weyl-type solutions and the Weyl-type matrices are studied. Section 5 is devoted to the solution of auxiliary inverse problems of recovering the potential on a fixed edge. In section 6 we study Inverse problem 1. For this inverse problem we provide a constructive procedure for the solution and prove its uniqueness.

Let us briefly explain the main ideas for constructing the solution of Inverse problem 1. On the first step, we solve auxiliary inverse problems (see Section 5) for each fixed $s = \overline{1,p} \setminus p_N$, and find the potential q_s on the edge e_s from the given $M_s(\lambda)$. For this purpose we develope the ideas of the method of spectral mappings [22, 23]. On the second step, using information on q_s, $s = \overline{1,p} \setminus p_N$, we construct the classical Weyl-type matrix $m_{p_N}(\lambda)$ related to the edge e_{p_N} . The last step is the classical one: we construct the potential q_{p_N} on the edge e_{p_N} from the given $m_{p_N}(\lambda)$ (this procedure was first described in [22, 23]).

3. Example. Let $m = 2$, $n_1 = 4$, $n_2 = 2$, and $0 = p_0 < p_1 < p_2 = p$. For simplicity, let $U_{j\nu}(y_j) = \gamma_{j\nu} y_j^{(\nu)}(l_j)$.

Case 1. Let $s = \overline{1, p_1}$. Then $k = 1, 2, 3$, and the boundary value problems $L_{sk\mu}$, $\mu = \overline{k, 3}$, are defined by the following boundary and matching conditions.

1) For $L_{s1\mu}$, $\mu = 1,2,3$:

$$y_s^{(\mu-1)}(0) = 0; \quad y_j(0) = y_j'(0) = y_j''(0) = 0,\ j = \overline{1,p_1} \setminus s; \qquad y_j(0) = 0,\ j = \overline{p_1+1,p_2},$$

$$\gamma_{p0} y_p(1) = \gamma_{j0} y_j(1),\ j = \overline{1,p-1}; \qquad \sum_{j=1}^{p} \gamma_{j1} y_j'(1) = 0,\ \sum_{j=1}^{p_1} \gamma_{j\nu} y_j^{(\nu)}(1) = 0,\ \nu = 2,3.$$

2) For $L_{s2\mu}$, $\mu = 2,3$:

$$y_s(0) = y_s^{(\mu-1)}(0) = 0; \quad y_j(0) = y_j'(0) = 0,\ j = \overline{1,p_1} \setminus s; \qquad y_j(0) = 0,\ j = \overline{p_1+1,p_2},$$

$$\gamma_{p0} y_p(1) = \gamma_{j0} y_j(1),\ j = \overline{1,p-1}; \quad \gamma_{p_1,1} y_{p_1}'(1) = \gamma_{j1} y_j'(1),\ j = \overline{1,p_1-1};$$

$$\sum_{j=1}^{p_1} \gamma_{j\nu} y_j^{(\nu)}(1) = 0,\ \nu = 2,3.$$

3) For $L_{s3\mu}$, $\mu = 3$:

$$y_s(0) = y_s'(0) = y_s''(0) = 0; \qquad y_j(0) = 0,\ j = \overline{1,p_1} \setminus s; \qquad y_j(0) = 0,\ j = \overline{p_1+1,p_2},$$

$$\gamma_{p0} y_p(1) = \gamma_{j0} y_j(1),\ j = \overline{1,p-1}; \quad \gamma_{p_1,\nu} y_{p_1}^{(\nu)}(1) = \gamma_{j\nu} y_j^{(\nu)}(1),\ j = \overline{1,p_1-1},\ \nu = 1,2;$$

$$\sum_{j=1}^{p_1} \gamma_{j3} y_j'''(1) = 0.$$

The Weyl-type solutions $\Psi_{sk} = \{\psi_{skj}\}$ are defined by the following boundary and matching conditions.

1) For Ψ_{s1} :

$$\psi_{s1s}(0,\lambda) = 1,$$

$$\psi_{s1j}(0,\lambda) = \psi_{s1j}'(0,\lambda) = \psi_{s1j}''(0,\lambda) = 0,\ j = \overline{1,p_1} \setminus s; \qquad \psi_{s1j}(0,\lambda) = 0,\ j = \overline{p_1+1,p_2},$$

$$\gamma_{p0} \psi_{s1p}(1,\lambda) = \gamma_{j0} \psi_{s1j}(1,\lambda),\ j = \overline{1,p-1};$$

$$\sum_{j=1}^{p} \gamma_{j1} \psi_{s1j}'(1,\lambda) = 0,\ \sum_{j=1}^{p_1} \gamma_{j\nu} \psi_{s1j}^{(\nu)}(1,\lambda) = 0,\ \nu = 2,3.$$

2) For Ψ_{s2} :

$$\psi_{s2s}(0,\lambda) = 0, \quad \psi_{s2s}'(0,\lambda) = 1,$$

$$\psi_{s2j}(0,\lambda) = \psi_{s2j}'(0,\lambda) = 0,\ j = \overline{1,p_1} \setminus s; \qquad \psi_{s2j}(0,\lambda) = 0,\ j = \overline{p_1+1,p_2},$$

$$\gamma_{p0} \psi_{s2p}(1,\lambda) = \gamma_{j0} \psi_{s2j}(1,\lambda),\ j = \overline{1,p-1}; \quad \gamma_{p_1,1} \psi_{s2p_1}'(1,\lambda) = \gamma_{j1} \psi_{s2j}'(1,\lambda),\ j = \overline{1,p_1-1};$$

$$\sum_{j=1}^{p_1} \gamma_{j\nu} \psi_{s2j}^{(\nu)}(1,\lambda) = 0,\ \nu = 2,3.$$

3) For Ψ_{s3} :

$$\psi_{s3s}(0,\lambda) = \psi_{s3s}'(0,\lambda) = 0, \quad \psi_{s3s}''(0,\lambda) = 1,$$

$$\psi_{s3j}(0,\lambda) = 0,\ j = \overline{1,p_1} \setminus s; \qquad \psi_{s3j}(0,\lambda) = 0,\ j = \overline{p_1+1,p_2},$$

$$\gamma_{p0} \psi_{s3p}(1,\lambda) = \gamma_{j0} \psi_{s3j}(1,\lambda),\ j = \overline{1,p-1};$$

$$\gamma_{p_1,\nu} \psi_{s3p_1}^{(\nu)}(1,\lambda) = \gamma_{j\nu} \psi_{s3j}^{(\nu)}(1,\lambda),\ j = \overline{1,p_1-1},\ \nu = 1,2;$$

$$\sum_{j=1}^{p_1} \gamma_{j3} \psi_{s3j}'''(1,\lambda) = 0.$$

Case 2. Let $s=\overline{p_1+1,p_2}$. Then $k=1$, and the boundary value problems $L_{s1\mu}$, $\mu=1,2$ are defined by the following boundary and matching conditions:

$$y_s^{(\mu-1)}(0)=0;\quad y_j(0)=y_j'(0)=y_j''(0)=0,\ j=\overline{1,p_1};\qquad y_j(0)=0,\ j=\overline{p_1+1,p_2}\setminus s,$$

$$\gamma_{p0}y_p(1)=\gamma_{j0}y_j(1),\ j=\overline{1,p-1};\qquad \sum_{j=1}^{p}\gamma_{j1}y_j'(1)=0.$$

The Weyl-type solutions $\Psi_{s1}=\{\psi_{s1j}\}$ are defined by the following boundary and matching conditions:

$$\psi_{s1s}(0,\lambda)=1,$$

$$\psi_{s1j}(0,\lambda)=\psi_{s1j}'(0,\lambda)=\psi_{s1j}''(0,\lambda)=0,\ j=\overline{1,p_1};\qquad \psi_{s1j}(0,\lambda)=0,\ j=\overline{p_1+1,p_2}\setminus s,$$

$$\gamma_{p0}\psi_{s1p}(1,\lambda)=\gamma_{j0}\psi_{s1j}(1,\lambda),\ j=\overline{1,p-1};\qquad \sum_{j=1}^{p}\gamma_{j1}\psi_{s1j}'(1,\lambda)=0.$$

4. Properties of spectral characteristics. Fix $i=\overline{1,m}$, $s=\overline{p_{i-1}+1,p_i}$. It follows from the boundary conditions (6) for the Weyl-type solutions that

$$\psi_{sks}(x_s,\lambda)=C_{ks}(x_s,\lambda)+\sum_{\mu=k+1}^{n_i}M_{sk\mu}(\lambda)C_{\mu s}(x_s,\lambda),\quad k=\overline{1,n_i}. \tag{10}$$

Using the fundamental system of solutions $\{C_{\mu j}(x_j,\lambda)\}$ on the edge e_j, one can write

$$\psi_{skj}(x_j,\lambda)=\sum_{\mu=1}^{n_l}M_{skj\mu}(\lambda)C_{\mu j}(x_j,\lambda),\ j=\overline{p_{l-1}+1,p_l},\ l=\overline{1,m},\ k=\overline{1,n_i-1}, \tag{11}$$

where the coefficients $M_{skj\mu}(\lambda)$ do not depend on x_j. In particular, $M_{sks\mu}(\lambda)=M_{sk\mu}(\lambda)$. Substituting (11) into boundary and matching conditions (6)-(9) for the Weyl-type solutions Ψ_{sk}, we obtain a linear algebraic system $D_{sk}^1(\lambda)$ with respect to $M_{skj\mu}(\lambda)$. Solving this system by Cramer's rule one gets $M_{skj\mu}(\lambda)=\Delta_{skj\mu}(\lambda)/\Delta_{sk}(\lambda)$, where the functions $\Delta_{skj\mu}(\lambda)$ and $\Delta_{sk}(\lambda)$ are entire in λ. Thus, the functions $M_{skj\mu}(\lambda)$ are meromorphic in λ, and consequently, the Weyl-type solutions and the Weyl-type matrices are meromorphic in λ. In particular,

$$M_{sk\mu}(\lambda)=\frac{\Delta_{sk\mu}(\lambda)}{\Delta_{sk}(\lambda)},\quad k<\mu, \tag{12}$$

where $\Delta_{sk\mu}(\lambda):=\Delta_{sks\mu}(\lambda)$. The function $\Delta_{sk\mu}(\lambda)$, $k\le\mu$ ($\Delta_{skk}(\lambda):=\Delta_{sk}(\lambda)$) is the characteristic function for the boundary value problem $L_{sk\mu}$, and its zeros coincide with the eigenvalues of $L_{sk\mu}$.

Fix $i=\overline{1,m}$. Let $\lambda=\rho_i^{n_i}$. The ρ_i – plane can be partitioned into sectors S of angle $\frac{\pi}{n_i}$ $\left(\arg\rho_i\in\left(\frac{\nu\pi}{n_i},\frac{(\nu+1)\pi}{n_i}\right),\ \nu=\overline{0,2n_i-1}\right)$ in which the roots $R_{i1},R_{i2},\dots,R_{i,n_i}$ of the equation $R^{n_i}-1=0$ can be numbered in such a way that

$$Re(\rho_iR_{i1})<Re(\rho_iR_{2i})<\ldots<Re(\rho_iR_{i,n_i}),\quad \rho_i\in S. \tag{13}$$

Let $\rho^*=\max\limits_{i=\overline{1,m}}\left(2n_i\max\limits_{\mu,j}\|q_{\mu j}\|_{L(0,l_j)}\right)$, $\mu=\overline{0,n_i-2}$, $j=\overline{p_{i-1}+1,p_i}$. It is known [27, Ch. 1] that for each fixed $j=\overline{p_{i-1}+1,p_i}$, on the edge e_j there exists a fundamental system of solutions of equation (1) $\{E_{kj}(x_j,\rho_i)\}_{k=\overline{1,n_i}}$ with the following properties.

1) For each sector S with property (13), the functions $E_{kj}^{(\nu-1)}(x_j,\rho_i)$, $\nu=\overline{1,n_i}$ are analytic in $\rho_i\in S$, $|\rho_i|>\rho^*$, and are continuous for $x_j\in[0,l_j]$, $\rho_i\in\overline{S}$, $|\rho_i|\ge\rho^*$;
2) As $|\rho_i|\to\infty$, $\rho_i\in\overline{S}$,

$$E_{kj}^{(\nu-1)}(x_j,\rho_i)=(\rho_iR_{ik})^{\nu-1}\exp(\rho_iR_{ik}x_j)[1], \tag{14}$$

where $k,\nu=\overline{1,n_i}$, $j=\overline{1,m}$, $[1]=1+O(\rho_i^{-1})$. The set of functions $\{E_{kj}(x_j,\rho_i)\}_{k=\overline{1,n_i}}$ is called the Birkhoff-type fundamental system of solutions on the edge e_j . Denote

$$\Omega_{ik}:=\det[R_{i\xi}^{\nu-1}]_{\xi,\nu=\overline{1,k}},\quad \Omega_{i0}:=1,\quad \omega_{ik}:=\frac{\Omega_{i,k-1}}{\Omega_{ik}},\quad k=\overline{1,n_i}.$$

Lemma 1. *Fix $i=\overline{1,m}$, $j=\overline{p_{i-1}+1,p_i}$, and fixed a sector S with property (13).*

1) Let $k=\overline{1,n_i-1}$, and let $y_j(x_j,\lambda)$ be a solution of equation (1) on the edge e_j under the conditions

$$y_j(0)=\ldots=y_j^{(k-1)}(0)=0. \tag{15}$$

Then for $x_j\in(0,l_j]$, $\nu=\overline{0,n_i-1}$, $\rho_i\in S,|\rho_i|\to\infty$,

$$y_j^{(\nu)}(x_j,\lambda)=\sum_{\mu=k+1}^{n_i}A_{\mu j}(\rho_i)(\rho_iR_{i\mu})^{\nu}\exp(\rho_iR_{i\mu}x_j)[1], \tag{16}$$

where the coefficients $A_{\mu j}(\rho_i)$ do not depend on x_j . Here and below we assume that $\arg\rho_i=$ const, when $|\rho_i|\to\infty$.

2) Let $k=\overline{1,n_i}$, and let $y_j(x_j,\lambda)$ be a solution of equation (1) on the edge e_j under the conditions

$$y_j(0)=\ldots=y_j^{(k-2)}(0)=0,\quad y_j^{(k-1)}(0)=1.$$

Then for $x_j\in(0,l_j]$, $\nu=\overline{0,n_i-1}$, $\rho_i\in S$, $|\rho_i|\to\infty$,

$$y_j^{(\nu)}(x_j,\lambda)=\frac{\omega_{ik}}{\rho_i^{k-1}}(\rho_iR_{ik})^{\nu}\exp(\rho_iR_{ik}x_j)[1]+\sum_{\mu=k+1}^{n_i}B_{\mu j}(\rho_i)(\rho_iR_{i\mu})^{\nu}\exp(\rho_iR_{i\mu}x_j)[1], \tag{17}$$

where the coefficients $B_{\mu j}(\rho_i)$ do not depend on x_j .

Proof. Using the fundamental system of solutions $\{E_{kj}(x_j,\rho_i)\}_{k=\overline{1,n_i}}$, one can write

$$y_j(x_j,\lambda)=\sum_{\mu=1}^{n_i}A_{\mu j}(\rho_i)E_{\mu j}(x_j,\rho_i). \tag{18}$$

Substituting (18) into (15) we obtain a linear algebraic system with respect to $A_{1j}(\rho_i),\ldots,A_{kj}(\rho_i)$. The determinant $A_k(\rho_i)$ of this system has the asymptotics $A_k(\rho_i)=\Omega_{ik}+O(\rho_i^{-1})$ as $|\rho_i|\to\infty$. Solving the system by Cramer's rule and taking (14) into account we get

$$A_{\xi j}(\rho_i)=\sum_{\mu=k+1}^{n_i}(c_{i\mu\xi j}+O(\rho_i^{-1}))A_{\mu j}(\rho_i),\quad \xi=\overline{1,k}, \tag{19}$$

where $c_{i\mu\xi j}$ are constants. Substituting (19) into (18) and using (14) we arrive at (16). Relations (17) are proved analogously. $\square$

Fix $i=\overline{1,m}$, $\xi=\overline{i,m}$, $s=\overline{p_{i-1}+1,p_i}$, $k=\overline{n_{\xi+1},n_\xi-1}$. Consider the following auxiliary linear algebraic system D_{sk}^0 with respect to the coefficients $B_{skj\mu}$:

$$\gamma_{p_l\nu}z_{skp_l\nu}-\gamma_{j\nu}z_{skj\nu}=0,\quad l=\overline{\xi+1,m},\ j=\overline{1,p_l-1},\ \nu=\overline{n_{l+1}-1,n_l-2},$$

$$\gamma_{p_\xi \nu} z_{skp_\xi \nu} - \gamma_{j\nu} z_{skj\nu} = 0, \quad j = \overline{1, p_\xi - 1}, \ \nu = \overline{n_{\xi+1} - 1, k-1},$$

$$\sum_{j=1}^{p_\xi} \gamma_{j\nu} z_{skj\nu} = 0, \ \nu = \overline{k, n_\xi - 1}, \quad \sum_{j=1}^{p_l} \gamma_{j\nu} z_{skj\nu} = 0, \ \nu = \overline{n_{l+1}, n_l - 1},$$

where

$$z_{sks\nu} = \sum_{\mu=k+1}^{n_i} B_{sks\mu} R_{i\mu}^{\nu}, \ z_{skj\nu} = \sum_{\mu=\max(n_l-k+1,2)}^{n_l} B_{skj\mu} R_{l\mu}^{\nu}, \quad j = \overline{p_{l-1}+1, p_l} \setminus s, \ l = \overline{1, m}.$$

Let d_{sk}^0 be the determinant of D_{sk}^0. We assume that $d_{sk}^0 \neq 0$ for each s, k and sectors S with property (13). This condition is called the regularity condition for matching. This is necessary for the existence and "regular behavior" of the Weyl-type solutions and the Weyl-type matrices (see [15] for details). Differential operators on T, which do not satisfy the regularity condition, possess qualitatively different properties for the formulation and the investigation of inverse problems, and are not considered in this paper; they require a separate investigation. We note that this condition is a generalization of the regularity condition for Sturm-Liouville operators on graphs (see [28]) and the regularity condition for higher-order differential operators on graphs [15]. In particular, for the so-called standard matching conditions [9], it is satisfied obviously.

Now we are going to study the asymptotic behavior of the Weyl-type solutions.

Lemma 2. *Fix $i = \overline{1, m}$, $s = \overline{p_{i-1}+1, p_i}$, and fix a sector S with property (13). For $x_s \in (0, l_s)$, $\nu = \overline{0, n_i - 1}$, $k = \overline{1, n_i}$, the following asymptotic formula holds*

$$\psi_{sks}^{(\nu)}(x_s, \lambda) = \frac{\omega_{ik}}{\rho_i^{k-1}} (\rho_i R_{ik})^{\nu} \exp(\rho_i R_{ik} x_s)[1], \quad \rho_i \in S, \ |\rho_i| \to \infty. \tag{20}$$

Proof. For $k = n_i$, (20) follows from Lemma 1. Fix $\xi = \overline{i, m}$, $k = \overline{n_{\xi+1}, n_\xi - 1}$. Using Lemma 1 and boundary conditions for Ψ_{sk} we get the following asymptotic formulae for $x_j \in (0, l_j)$, $|\lambda| \to \infty$ inside the corresponding sectors:

$$\psi_{sks}^{(\nu)}(x_s, \lambda) = \frac{\omega_{ik}}{\rho_i^{k-1}} (\rho_i R_{ik})^{\nu} \exp(\rho_i R_{ik} x_s)[1] + \sum_{\mu=k+1}^{n_i} A_{sks\mu}(\rho_i)(\rho_i R_{i\mu})^{\nu} \exp(\rho_i R_{i\mu} x_s)[1], \tag{21}$$

$$\psi_{skj}^{(\nu)}(x_j, \lambda) = \sum_{\mu=\max(n_l-k+1,2)}^{n_l} A_{skj\mu}(\rho_l)(\rho_l R_{l\mu})^{\nu} \exp(\rho_l R_{l\mu} x_j)[1], \ j = \overline{p_{l-1}+1, p_l}, \ l = \overline{1, m} \setminus s, \tag{22}$$

Substituting (21), (22) into matching conditions (8)-(9) for Ψ_{sk}, we obtain a linear algebraic system $D_{sk}(\lambda)$ with respect to the coefficients $A_{skj\mu}$. The determinant $d_{sk}(\lambda)$ of this system has the asymptotics

$$d_{sk}(\lambda) = d_{sk}^0 \exp\Big(\sum_{l=1}^{m} \rho_l \alpha_{lsk}\Big)[1], \ |\lambda| \to \infty, \tag{23}$$

inside the corresponding sectors, where α_{lsk} are constants depending on $l_1, \ldots, l_p$, namely:

$$\alpha_{isk} := \Big(\sum_{\mu=k+1}^{n_i} R_{i\mu}\Big) l_s + \Big(\sum_{\mu=n_i-k+1}^{n_i} R_{i\mu}\Big)\Big(\sum_{j=p_{i-1}+1}^{p_i} l_j - l_s\Big),$$

$$\alpha_{lsk} := \Big(\sum_{\mu=\max(n_l-k+1,2)}^{n_l} R_{l\mu}\Big)\Big(\sum_{j=p_{l-1}+1}^{p_l} l_j\Big), \ l \neq i.$$

Solving the system $D_{sk}(\lambda)$ by Cramer's rule and using (23), we obtain in particular,

$$A_{sks\mu}(\rho) = O(\rho_i^{1-k}\exp(\rho_i(R_{ik} - R_{i\mu})l_s)), \quad k = \overline{1, n_i - 1}, \ \mu = \overline{k+1, n_i}. \tag{24}$$

Substituting (24) into (21) we arrive at (20). $\square$

It follows from the proof of Lemma 2 that one can also get the asymptotics for $\psi_{skj}^{(\nu)}(x_j, \lambda)$, $j \neq s$; but for our purposes only (20) is needed.

4. Auxiliary inverse problems. In this section we consider auxiliary inverse problems of recovering differential operator on each fixed edge. Fix $s = \overline{1, p}$, and consider the following inverse problem on the edge e_s .

Inverse problem 2. Given the Weyl-type matrix M_s , construct the potential q_s on the edge e_s .

In this inverse problem we construct the potential only on the edge e_s , but the Weyl-type matrix M_s brings a global information from the whole graph. In other words, this problem is not a local inverse problem related only to the edge e_s .

Let us prove the uniqueness theorem for the solution of Inverse problem 2. For this purpose together with q we consider a potential $\tilde{q}$. Everywhere below if a symbol α denotes an object related to q, then $\tilde{\alpha}$ will denote the analogous object related to $\tilde{q}$.

Theorem 1. *Fix $s = \overline{1, p}$. If $M_s = \tilde{M}_s$, then $q_s = \tilde{q}_s$. Thus, the specification of the Weyl-type matrix M_s uniquely determines the potential q_s on the edge e_s .*

We omit the proof since it is similar to that in [23, Ch.2]. Moreover, using the method of spectral mappings and the asymptotics (20) for the Weyl-type solutions, one can get a constructive procedure for the solution of Inverse problem 2. It can be obtained by the same arguments as for n-th order differential operators on a finite interval (see [23, Ch.2] for details). Note that like in [23], the nonlinear Inverse problem 2 is reduced to the solution of a linear equation in the corresponding Banach space of sequences. The unique solvability of this linear equation is proved by the same arguments as in [23].

Fix $i = \overline{1, m}$, $j = \overline{p_{i-1} + 1, p_i}$. Now we define an auxiliary Weyl-type matrix with respect to the internal vertex v_0 and the edge e_j .

Let $\varphi_{jk}(x_j, \lambda)$, $k = \overline{1, n_i}$, be solutions of equation (1) on the edge e_j under the conditions

$$\varphi_{jk}^{(\nu-1)}(l_j, \lambda) = \delta_{k\nu}, \ \nu = \overline{1, k}, \quad \varphi_{jk}^{(\mu-1)}(0, \lambda) = 0, \ \mu = \overline{1, n_i - k}.$$

We introduce the matrix $m_j(\lambda) = [m_{jk\nu}(\lambda)]_{k,\nu=\overline{1,n_i}}$, where $m_{jk\nu}(\lambda) := \varphi_{jk}^{(\nu-1)}(l_j, \lambda)$. Clearly, $m_{jk\nu}(\lambda) = \delta_{k\nu}$ for $k \geq \nu$, and $\det m_j(\lambda) \equiv 1$. The matrix $m_j(\lambda)$ is called the Weyl-type matrix with respect to the internal vertex v_0 and the edge e_j . Consider the following inverse problem on the edge e_j .

Inverse problem 3. Fix $j = \overline{1, p}$. Given the Weyl-type matrix m_j , construct the potential q_j on the edge e_j .

This inverse problem is the classical one, since it is the inverse problem of recovering a higher-order differential equation on a finite interval from its Weyl-type matrix. This inverse problem has been solved in [23], where the uniqueness theorem for this inverse problem is proved. Moreover, in [23] an algorithm for the solution of Inverse problem 3 is given, and necessary and sufficient conditions for the solvability of this inverse problem are provided.

5. Solution of Inverse Problem 1. In this section we obtain a constructive procedure for the solution of Inverse problem 1 and prove their uniqueness. First we prove an auxiliary assertion.

Lemma 3. *Fix* $i=\overline{1,m}$, $j=\overline{p_{i-1}+1,p_i}$. *Then for each fixed* $s=\overline{1,p_1}\setminus j$,

$$m_{j1\nu}(\lambda)=\frac{\psi_{s1j}^{(\nu-1)}(l_j,\lambda)}{\psi_{s1j}(l_j,\lambda)},\quad \nu=\overline{2,n_i}, \tag{25}$$

$$m_{jk\nu}(\lambda)=\frac{\det[\psi_{s\mu j}(l_j,\lambda),\ldots,\psi_{s\mu j}^{(k-2)}(l_j,\lambda),\psi_{s\mu j}^{(\nu-1)}(l_j,\lambda)]_{\mu=\overline{1,k}}}{\det[\psi_{s\mu j}^{(\xi-1)}(l_j,\lambda)]_{\xi,\mu=\overline{1,k}}},\ 2\le k<\nu\le n_i. \tag{26}$$

Proof. Denote

$$w_{js}(x_j,\lambda):=\frac{\psi_{s1j}(x_j,\lambda)}{\psi_{s1j}(l_j,\lambda)}.$$

The function $w_{js}(x_j,\lambda)$ is a solution of equation (1) on the edge e_j, and $w_{js}(l_j,\lambda)=1$. Moreover, by virtue of the boundary conditions on Ψ_{s1}, one has $w_{js}^{(\xi-1)}(0,\lambda)=0$, $\xi=\overline{1,n_i-1}$. Hence, $w_{js}(x_j,\lambda)\equiv\varphi_{j1}(x_j,\lambda)$, i.e.

$$\varphi_{j1}(x_j,\lambda)=\frac{\psi_{s1j}(x_j,\lambda)}{\psi_{s1j}(l_j,\lambda)}. \tag{27}$$

Similarly, we calculate

$$\varphi_{jk}(x_j,\lambda)=\frac{\det[\psi_{s\mu j}(l_j,\lambda),\ldots,\psi_{s\mu j}^{(k-2)}(l_j,\lambda),\psi_{s\mu j}(x_j,\lambda)]_{\mu=\overline{1,k}}}{\det[\psi_{s\mu j}^{(\xi-1)}(l_j,\lambda)]_{\xi,\mu=\overline{1,k}}},\quad k=\overline{2,n_i-1}. \tag{28}$$

Since $m_{jk\nu}(\lambda)=\varphi_{jk}^{(\nu-1)}(l_j,\lambda)$, it follows from (27)-(28) that (25)-(26) hold. □

Now we are going to obtain a constructive procedure for the solution of Inverse problem 1. Our plan is the following.

Step 1. Let the Weyl-type matrices $\{M_s(\lambda)\}$, $s=\overline{1,p}\setminus p_N$, be given. Solving Inverse problem 2 for each fixed $s=\overline{1,p}\backslash p_N$, we find the potentials q_s on the edges e_s, $s=\overline{1,p}\backslash p_N$.

Step 2. Using the knowledge of the potential on the edges e_s, $s=\overline{1,p}\backslash p_N$, we construct the Weyl-type matrix m_{p_N}.

Step 3. Solving Inverse problem 3 for $j=p_N$ we find the potential q_{p_N} on the edge e_{p_N}.

Steps 1 and 3 have been already studied in Section 4. It remains to fulfil Step 2.

Suppose that Step 1 was already made, and we found the potentials q_s, $s=\overline{1,p}\setminus p_N$, on the edges e_s, $s=\overline{1,p}\setminus p_N$. Then we calculate the functions $C_{kj}(x_j,\lambda)$, $j=\overline{1,p}\setminus p_N$; here $k=\overline{1,n_i}$ for $j=\overline{p_{i-1}+1,p_i}$.

Fix $s=\overline{1,p_1}$ (if $N>1$), and $s=\overline{1,p_1-1}$ (if $N=1$). All calculations below will be made for this fixed s.

Our goal now is to construct the Weyl-type matrix $m_{p_N}(\lambda)$. For this purpose we will use Lemma 3. According to (25)-(26), in order to construct $m_{p_N}(\lambda)$ we have to calculate the functions

$$\psi_{skp_N}^{(\nu)}(l_{p_N},\lambda),\quad k=\overline{1,n_N-1},\ \nu=\overline{0,n_N-1}. \tag{29}$$

We will find the functions (29) by the following steps.

1) Using (10) we construct the functions

$$\psi_{sks}^{(\nu)}(l_s,\lambda),\ k=\overline{1,n_N-1},\ \nu=\overline{0,n_1-1}, \tag{30}$$

by the formula

$$\psi_{sks}^{(\nu)}(l_s,\lambda)=C_{ks}^{(\nu)}(l_s,\lambda)+\sum_{\mu=k+1}^{n_1}M_{sk\mu}(\lambda)C_{\mu s}^{(\nu)}(l_s,\lambda). \tag{31}$$

Clearly, one can construct (30) for $k=\overline{1,n_1-1}$, but we need (30) only for $k=\overline{1,n_N-1}$.

2) Consider a part of the matching conditions (8) on Ψ_{sk}. More precisely, let $\xi=\overline{N,m}$, $k=\overline{n_{\xi+1},n_\xi-1}$, $l=\overline{\xi,m}$, $j=\overline{1,p_l-1}$. Then, in particular, (8) yields

$$U_{p_l,\nu}(\psi_{skp_l})=U_{j\nu}(\psi_{skj}),\quad \nu=\overline{n_{l+1}-1,\min(k-1,n_l-2)}. \tag{32}$$

Since the functions (30) are known, it follows from (32) that one can calculate the functions

$$\psi_{skj}^{(\nu)}(l_j,\lambda),\ \xi=\overline{N,m},\ k=\overline{n_{\xi+1},n_\xi-1},\ l=\overline{\xi,m},\ j=\overline{1,p_l},\ \nu=\overline{n_{l+1}-1,\min(k-1,n_l-2)}. \tag{33}$$

In particular we found the functions (29) for $\nu=\overline{0,k-1}$.

3) It follows from (11) and the boundary conditions on Ψ_{sk} that

$$\psi_{skj}(x_j,\lambda)=\sum_{\mu=\max(n_l-k+1,2)}^{n_l}M_{skj\mu}(\lambda)C_{\mu j}(x_j,\lambda),\ k=\overline{1,n_1-1},\ l=\overline{1,m},\ j=\overline{p_{l-1}+1,p_l}\setminus s,$$

and consequently,

$$\psi_{skj}^{(\nu)}(l_j,\lambda)=\sum_{\mu=\max(n_l-k+1,2)}^{n_l}M_{skj\mu}(\lambda)C_{\mu j}^{(\nu)}(l_j,\lambda), \tag{34}$$

$$k=\overline{1,n_1-1},\ l=\overline{1,m},\ j=\overline{p_{l-1}+1,p_l}\setminus s,\ \nu=\overline{0,n_l-1}.$$

We consider only a part of relations (34). More precisely, let $\xi=\overline{N,m}$, $k=\overline{n_{\xi+1},n_\xi-1}$, $l=\overline{1,m}$, $j=\overline{p_{l-1}+1,p_l}$, $j\neq p_N$, $j\neq s$, $\nu=\overline{0,\min(k-1,n_l-2)}$. Then

$$\sum_{\mu=\max(n_l-k+1,2)}^{n_l}M_{skj\mu}(\lambda)C_{\mu j}^{(\nu)}(l_j,\lambda)=\psi_{skj}^{(\nu)}(l_j,\lambda),\quad \nu=\overline{0,\min(k-1,n_l-2)}. \tag{35}$$

For this choice of parameters, the right-hand side in (35) are known, since the functions (33) are known. Relations (35) form a linear algebraic system σ_{skj} with respect to the coefficients $M_{skj\mu}(\lambda)$. Solving the system by Cramer's rule we find the functions $M_{skj\mu}(\lambda)$. Substituting them into (34), we calculate the functions

$$\psi_{skj}^{(\nu)}(l_j,\lambda),\quad k=\overline{1,n_N-1},\ l=\overline{1,m},\ j=\overline{p_{l-1}+1,p_l}\setminus p_N,\ \nu=\overline{0,n_l-1}. \tag{36}$$

Note that for $j=s$ these functions were found earlier.

4) Let us now use the generalized Kirchhoff'sconditions (9) for Ψ_{sk}. Since the functions (36) are known, one can construct by (9) the functions (29) for $k=\overline{1,n_N-1}$, $\nu=\overline{k,n_N-1}$. Thus, the functions (29) are known for $k=\overline{1,n_N-1}$, $\nu=\overline{0,n_N-1}$.

Since the functions (29) are known, we construct the Weyl-type matrix $m_{p_N}(\lambda)$ via (25)-(26) for $j=p_N$. Thus, we have obtained the solution of Inverse problem 1 and proved its uniqueness, i.e. the following assertion holds.

Theorem 2. *The specification of the Weyl-type matrices $M_s(\lambda)$, $s=\overline{1,p}\setminus p_N$, uniquely determines the potential q on T. The solution of Inverse problem 1 can be obtained by the following algorithm.*

Algorithm 1. *Given the Weyl-type matrices $M_s(\lambda)$, $s=\overline{1,p}\setminus p_N$.*

1) Find the potentials q_s, $s=\overline{1,p}\setminus p_N$, by solving Inverse problem 2 for each fixed $s=\overline{1,p}\setminus p_N$.

2) Calculate $C_{kj}^{(\nu)}(l_j,\lambda)$, $j=\overline{1,p}\setminus p_N$; here $k=\overline{1,n_i}$, $\nu=\overline{0,n_i-1}$ for $j=\overline{p_{i-1}+1,p_i}$.

3) Fix $s=\overline{1,p_1}$ (if $N>1$), and $s=\overline{1,p_1-1}$ (if $N=1$). All calculations below will be made for this fixed s. Construct the functions (30) via (31).

4) Calculate the functions (33) using (32).
5) Find the functions $M_{skj\mu}(\lambda)$, *by solving the linear algebraic systems* σ_{skj} .
6) Construct the functions (29) using (9).
7) Calculate the Weyl-type matrix $m_{p_N}(\lambda)$ *via (25)-(26) for* $j = p_N$.
8) Construct the potential q_{p_N} *on the edge* e_{p_N} *by solving Inverse problem 3.*

Remark 1. *Inverse problem from a system of spectra.* The zeros $\Lambda_{sk\mu} := \{\lambda_{lsk\mu}\}_{l\geq 1}$ of the entire functions $\Delta_{sk\mu}(\lambda)$ coincide with the eigenvalues of the boundary value problems $L_{sk\mu}$. The inverse problems of recovering the potential q from systems of spectra are formulated as follows.

Inverse problem 4. Given $\Lambda_{sk\mu}$, $s = \overline{1,p} \setminus p_N$, $k \leq \mu$, construct q on T.

Since the functions $\Delta_{sk\mu}(\lambda)$ are uniquely determined by their zeros, it follows from (12) that this inverse problem can be reduced to Inverse problem 1.

Acknowledgment. This research was supported in part by DAAD and by Grant 13-01-00134 of Russian Foundation for Basic Research.

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Name:	Yurko, Vjacheslav Anatoljevich
Place of work:	Department of Mathematics, Saratov University Astrakhanskaya 83, Saratov 410012, Russia
E-mail:	yurkova@info.sgu.ru

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