

SCHRIFTENREIHE DER FAKULTÄT FÜR MATHEMATIK

On the Maxwell Inequalities
for Bounded and Convex Domains

by

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Dedicated with highest gratitude to my academic teachers
Norbert Weck and Karl-Josef Witsch
on the occasion of their 70th resp. 65th birthday

Abstract

For a bounded and convex domain in three dimensions we show that the Maxwell constants are bounded from below and above by Friedrichs' and Poincaré's constants.

Key Words Maxwell's equations, Maxwell constant, second Maxwell eigenvalue, electro statics, magneto statics, Poincaré's inequality, Friedrichs' inequality, Poincaré's constant, Friedrichs' constant

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1 Introduction

Let $\Omega \subset \mathbb{R}^3$ be a bounded and convex domain. It is well known that, e.g., by Rellich's selection theorem using standard indirect arguments, the Poincaré* inequalities

$$\exists c_{p,o} > 0 \quad \forall u \in \mathring{H}^1 \quad |u| \leq c_{p,o} |\nabla u|, \quad (1.1)$$

$$\exists c_p > 0 \quad \forall u \in H^1 \cap \mathbb{R}^\perp \quad |u| \leq c_p |\nabla u| \quad (1.2)$$

hold. Here, $c_{p,o}$ and c_p are the Poincaré constants, which satisfy

$$0 < c_{p,o} = 1/\sqrt{\lambda_1} < 1/\sqrt{\mu_2} = c_p,$$

*The estimate (1.1) is often called Friedrichs'/Steklov inequality as well.

where λ_1 is the first Dirichlet and μ_2 the second Neumann eigenvalue of the Laplacian. By $\langle \cdot, \cdot \rangle$ and $|\cdot|$ we denote the standard inner product and induced norm in L^2 and we will write the usual L^2 -Sobolev spaces as H^1 and $\overset{\circ}{H}^1$, the latter is defined as the closure in H^1 of smooth and compactly supported test functions. All spaces and norms are defined on Ω . Moreover, we introduce the standard Sobolev spaces for the rotation and divergence by R and D . As before, we will denote the closures of test vector fields in the respective graph norms by $\overset{\circ}{R}$ and $\overset{\circ}{D}$. An index zero at the lower right corner of the latter spaces indicates a vanishing derivative, e.g.,

$$R_0 := \{E \in R : \operatorname{rot} E = 0\}, \quad \overset{\circ}{D}_0 := \{E \in \overset{\circ}{D} : \operatorname{div} E = 0\}.$$

As Ω is convex, it is especially simply connected and has got a connected boundary. Hence, the Neumann and Dirichlet fields of Ω vanish, i.e., $R_0 \cap \overset{\circ}{D}_0 = \overset{\circ}{R}_0 \cap R_0 = \{0\}$. By the Maxwell compactness properties, i.e., the compactness of the two embeddings

$$\overset{\circ}{R} \cap D \hookrightarrow L^2, \quad R \cap \overset{\circ}{D} \hookrightarrow L^2,$$

(and again by a standard indirect argument) the Maxwell inequalities

$$\exists c_{m,t} > 0 \quad \forall E \in \overset{\circ}{R} \cap D \quad |E| \leq c_{m,t} (|\operatorname{rot} E|^2 + |\operatorname{div} E|^2)^{1/2}, \quad (1.3)$$

$$\exists c_{m,n} > 0 \quad \forall H \in R \cap \overset{\circ}{D} \quad |H| \leq c_{m,n} (|\operatorname{rot} H|^2 + |\operatorname{div} H|^2)^{1/2} \quad (1.4)$$

hold. To the best of the author's knowledge, general bounds for the Maxwell constants $c_{m,t}$ and $c_{m,n}$ are missing. On the other hand, at least estimates for $c_{m,t}$ and $c_{m,n}$ from above are very important from the point of view of applications, such as preconditioning or a priori and a posteriori error estimation for numerical methods.

In the paper at hand we will prove that

$$c_{p,o} \leq c_{m,t}, c_{m,n} \leq c_p \leq \operatorname{diam}(\Omega)/\pi \quad (1.5)$$

holds true. We note that (1.5) is already well known in two dimensions, even for general Lipschitz domains $\Omega \subset \mathbb{R}^2$ (except of the last inequality), but new in three dimensions. Furthermore, the last inequality in (1.5) has been proved in the famous paper of Payne and Weinberger [9], where also the optimality of the estimate was shown. This paper contains a small mistake, which has been corrected in [2].

2 Results and Proofs

We start with an inequality for irrotational fields.

Lemma 1 *For all $E \in \nabla \overset{\circ}{H}^1 \cap D$ and all $H \in \nabla H^1 \cap \overset{\circ}{D}$*

$$|E| \leq c_{p,o} |\operatorname{div} E|, \quad |H| \leq c_p |\operatorname{div} H|.$$

Proof Let $\varphi \in \mathring{H}^1$ with $E = \nabla\varphi$. By (1.1) we get

$$|E|^2 = \langle E, \nabla\varphi \rangle = -\langle \operatorname{div} E, \varphi \rangle \leq |\operatorname{div} E| |\varphi| \leq c_{p,o} |\operatorname{div} E| |\nabla\varphi| = c_{p,o} |\operatorname{div} E| |E|.$$

Let $\varphi \in H^1$ with $H = \nabla\varphi$ and $\varphi \perp \mathbb{R}$. Since $H \in \mathring{D}$ and by (1.2) we obtain

$$|H|^2 = \langle H, \nabla\varphi \rangle = -\langle \operatorname{div} H, \varphi \rangle \leq |\operatorname{div} H| |\varphi| \leq c_p |\operatorname{div} H| |\nabla\varphi| = c_p |\operatorname{div} H| |H|,$$

completing the proof. \square

Remark 2 Clearly, Lemma 1 extends to arbitrary Lipschitz domains $\Omega \subset \mathbb{R}^N$, $N \in \mathbb{N}$.

As usual in the theory of Maxwell's equations, we need another crucial tool, the Helmholtz decompositions of vector fields into irrotational and solenoidal vector fields. For convex domains, these decompositions are very simple. We have

$$\mathbf{L}^2 = \nabla\mathring{H}^1 \oplus \operatorname{rot} \mathbf{R}, \quad \mathbf{L}^2 = \nabla H^1 \oplus \operatorname{rot} \mathring{R}, \quad (2.1)$$

where \oplus denotes the orthogonal sum in \mathbf{L}^2 . We note

$$\mathring{R}_0 = \nabla\mathring{H}^1, \quad R_0 = \nabla H^1, \quad D_0(\Omega) = \operatorname{rot} \mathbf{R}, \quad \mathring{D}_0 = \operatorname{rot} \mathring{R}.$$

Moreover, with

$$\mathring{\mathcal{R}} := \mathring{R} \cap \operatorname{rot} \mathbf{R}, \quad \mathcal{R} := \mathbf{R} \cap \operatorname{rot} \mathring{R}$$

we have

$$\mathring{R} = \nabla\mathring{H}^1 \oplus \mathring{\mathcal{R}}, \quad \mathbf{R} = \nabla H^1 \oplus \mathcal{R} \quad (2.2)$$

and see

$$\operatorname{rot} \mathring{R} = \operatorname{rot} \mathring{\mathcal{R}}, \quad \operatorname{rot} \mathbf{R} = \operatorname{rot} \mathcal{R}.$$

We note that all occurring spaces are closed subspaces of \mathbf{L}^2 , which follows immediately by the estimates (1.1)-(1.4). More details about the Helmholtz decompositions can be found e.g. in [6].

To get similar inequalities for solenoidal vector fields as in Lemma 1 we need a crucial lemma from [1, Theorem 2.17], see also [10, 5, 4, 3] for related partial results.

Lemma 3 Let E belong to $\mathring{R} \cap \mathring{D}$ or $\mathbf{R} \cap \mathring{D}$. Then $E \in H^1$ and

$$|\nabla E|^2 \leq |\operatorname{rot} E|^2 + |\operatorname{div} E|^2. \quad (2.3)$$

We emphasize that for $E \in \mathring{H}^1$ and any domain $\Omega \subset \mathbb{R}^3$

$$|\nabla E|^2 = |\operatorname{rot} E|^2 + |\operatorname{div} E|^2 \quad (2.4)$$

holds since $-\Delta = \operatorname{rot} \operatorname{rot} - \nabla \operatorname{div}$. This formula is no longer valid if E has just the tangential or normal boundary condition but for convex domains the inequality (2.3) remains true.

Lemma 4 For all vector fields E in $\mathring{R} \cap \text{rot } R$ or $R \cap \text{rot } \mathring{R}$

$$|E| \leq c_p |\text{rot } E|.$$

Proof Let $E \in \text{rot } R = \text{rot } \mathcal{R}$ and $\Phi \in \mathcal{R}$ with $\text{rot } \Phi = E$. Then $\Phi \in H^1$ by Lemma 3 since $\mathcal{R} = R \cap \mathring{D}_0$. Moreover, $\Phi = \text{rot } \Psi$ can be represented by some $\Psi \in \mathring{R}$. Hence, for any constant vector $a \in \mathbb{R}^3$ we have $\langle \Phi, a \rangle = 0$. Thus, Φ belongs to $H^1 \cap (\mathbb{R}^3)^\perp$. Then, since $E \in \mathring{R}$ and by Lemma 3 we get

$$|E|^2 = \langle E, \text{rot } \Phi \rangle = \langle \text{rot } E, \Phi \rangle \leq |\text{rot } E| |\Phi| \leq c_p |\text{rot } E| |\nabla \Phi| \leq c_p |\text{rot } E| \underbrace{|\text{rot } \Phi|}_{=E}.$$

If $E \in \text{rot } \mathring{R}$ there exists $\Phi \in \mathring{R}$ with $\text{rot } \Phi = E$. Using (2.2) we decompose

$$E = E_0 + E_{\text{rot}} \in R_0 \oplus \mathcal{R}.$$

Then, $\text{rot } E_{\text{rot}} = \text{rot } E$ and again by Lemma 3 we see $E_{\text{rot}} \in H^1$. Let $a \in \mathbb{R}^3$ such that $E_{\text{rot}} - a \in H^1 \cap (\mathbb{R}^3)^\perp$. Since $\Phi \in \mathring{R}$, $\langle \text{rot } \Phi, H_0 \rangle$ and $\langle \text{rot } \Phi, a \rangle$ vanish. By Lemma 3

$$|E|^2 = \langle \text{rot } \Phi, E \rangle = \underbrace{\langle \text{rot } \Phi, E_{\text{rot}} - a \rangle}_{=E} \leq |E| |E_{\text{rot}} - a| \leq c_p |E| |\nabla E_{\text{rot}}| \leq c_p |E| \underbrace{|\text{rot } E_{\text{rot}}|}_{=\text{rot } E}$$

holds, which completes the proof. \square

Remark 5 It is well known that Lemma 4 holds in two dimensions for any Lipschitz domain $\Omega \subset \mathbb{R}^2$. This follows immediately from Lemma 1 if we take into account that in two dimensions the rotation rot is given by the divergence div after 90° -rotation of the vector field to which it is applied.

Theorem 6 For all vector fields $E \in \mathring{R} \cap D$ and $H \in R \cap \mathring{D}$

$$|E|^2 \leq c_{p,0}^2 |\text{div } E|^2 + c_p^2 |\text{rot } E|^2, \quad |H|^2 \leq c_p^2 |\text{div } H|^2 + c_p^2 |\text{rot } H|^2$$

hold, i.e., $c_{m,t}, c_{m,n} \leq c_p$. Moreover, $c_{p,0} \leq c_{m,t}, c_{m,n} \leq c_p \leq \text{diam}(\Omega)/\pi$.

Proof By the Helmholtz decomposition (2.1) we have

$$\mathring{R} \cap D \ni E = E_\nabla + E_{\text{rot}} \in \nabla \mathring{H}^1 \oplus \text{rot } R$$

with $E_\nabla \in \nabla \mathring{H}^1 \cap D$ and $E_{\text{rot}} \in \mathring{R} \cap \text{rot } R$ as well as $\text{div } E_\nabla = \text{div } E$ and $\text{rot } E_{\text{rot}} = \text{rot } E$. By Lemma 1 and Lemma 4 and orthogonality we obtain

$$|E|^2 = |E_\nabla|^2 + |E_{\text{rot}}|^2 \leq c_{p,0}^2 |\text{div } E|^2 + c_p^2 |\text{rot } E|^2.$$

Similarly we have

$$R \cap \mathring{D} \ni H = H_\nabla + H_{\text{rot}} \in \nabla H^1 \oplus \text{rot } \mathring{R}$$

with $H_{\nabla} \in \mathbf{H}^1 \cap \mathring{\mathbf{D}}$ and $H_{\text{rot}} \in \mathbf{R} \cap \text{rot } \mathring{\mathbf{R}}$ as well as $\text{div } H_{\nabla} = \text{div } H$ and $\text{rot } H_{\text{rot}} = \text{rot } H$. As before,

$$|H|^2 = |H_{\nabla}|^2 + |H_{\text{rot}}|^2 \leq c_p^2 |\text{div } H|^2 + c_p^2 |\text{rot } H|^2.$$

This shows the upper bound. For the lower bound we look at (1.3) and (1.4) and obtain

$$\frac{1}{c_{\text{m,t}}^2} = \inf_{0 \neq E \in \mathring{\mathbf{R}} \cap \mathring{\mathbf{D}}} \frac{|\text{rot } E|^2 + |\text{div } E|^2}{|E|^2}, \quad \frac{1}{c_{\text{m,n}}^2} = \inf_{0 \neq E \in \mathbf{R} \cap \mathring{\mathbf{D}}} \frac{|\text{rot } E|^2 + |\text{div } E|^2}{|E|^2}.$$

Thus

$$\frac{1}{c_{\text{m,t}}^2}, \frac{1}{c_{\text{m,n}}^2} \leq \inf_{0 \neq E \in \mathbf{H}^1} \frac{|\text{rot } E|^2 + |\text{div } E|^2}{|E|^2} = \inf_{0 \neq E \in \mathring{\mathbf{H}}^1} \frac{|\nabla E|^2}{|E|^2} = \frac{1}{c_{\text{p,0}}^2},$$

completing the proof. \square

Remark 7

- (i) *It is unclear but most probable that $c_{\text{p,0}} < c_{\text{m,t}} < c_{\text{m,n}} = c_{\text{p}}$ holds. In forthcoming publications [7, 8] we will show more and sharper estimates on the Maxwell constants, showing additional and sharp relations between the Maxwell and the Poincaré/Friedrichs/Steklov constants.*
- (ii) *Our results extend also to all polyhedrons which allow the \mathbf{H}^1 -regularity of the Maxwell spaces $\mathring{\mathbf{R}} \cap \mathring{\mathbf{D}}$ and $\mathbf{R} \cap \mathring{\mathbf{D}}$ or to domains whose boundaries consist of combinations of convex boundary parts and polygonal parts which allow the \mathbf{H}^1 -regularity. It is shown in [3, Theorem 4.1] that (2.3), even (2.4), still holds for all $E \in \mathbf{H}^1 \cap \mathring{\mathbf{R}}$ or $E \in \mathbf{H}^1 \cap \mathring{\mathbf{D}}$ if Ω is a polyhedron[†]. We note that even some non-convex polyhedrons admit the \mathbf{H}^1 -regularity of the Maxwell spaces depending on the angle of the corners, which are not allowed to be too pointy.*

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[†]The crucial point is that the unit normal is piecewise constant.

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