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An Elementary Extension of Korn's First Inequality to H(Curl) motivated by Gradient Plasticity with Plastic Spin

by

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An Elementary Extension of Korn's First Inequality to $\mathsf{H}(\mathrm{Curl})$ motivated by Gradient Plasticity with Plastic Spin

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Dedicated to Professor Rolf Leis
on the occasion of his 80th birthday

Abstract

We prove a Korn-type inequality in $\overset{\circ}{\mathsf{H}}(\mathrm{Curl};\Omega)$ for non-symmetric tensor fields P mapping Ω to $\mathbb{R}^{3\times 3}$. More precisely, let $\Omega\subset\mathbb{R}^3$ be a simply connected and bounded domain with Lipschitz boundary $\partial\Omega$. Then, there exists a constant $c>0$ such that

$$c\,\|P\|_{\mathsf{L}^2(\Omega)}\leq\|\operatorname{sym}P\|_{\mathsf{L}^2(\Omega)}+\|\operatorname{Curl}P\|_{\mathsf{L}^2(\Omega)}\tag{0.1}$$

holds for all tensor fields $P\in\overset{\circ}{\mathsf{H}}(\mathrm{Curl};\Omega)$, i.e., all $P\in\mathsf{H}(\mathrm{Curl};\Omega)$ with vanishing tangential trace $\nu\times P$ on $\partial\,\Omega$. Here rotation and tangential trace are defined row-wise and ν denotes the outward unit normal for $\partial\,\Omega$. For compatible $P=\nabla v$ with vector fields $v\in\mathsf{H}^1(\Omega)$ and $\nu\times\nabla v=0$ on $\partial\Omega$ the former reduces to a non-standard variant of Korn's first inequality

$$c\,\|\nabla v\|_{\mathsf{L}^2(\Omega)}\leq\|\operatorname{sym}\nabla v\|_{\mathsf{L}^2(\Omega)}\,.$$

Key Words Korn's inequality, gradient plasticity, micromorphic model, Maxwell's equations, Helmholtz decomposition, Poincaré type estimate

1 Introduction

The motivation of our investigation is a formulation of infinitesimal gradient plasticity in the theory of constitutive equations with internal variables to describe the irreversible deformation behaviour of metals at small strain [1, 9, 5, 6]. The finite strain case has been dealt with in [15]. Our model is taken from Neff et al. [21]. Let $\Omega\subset\mathbb{R}^3$ be a bounded domain. A simplified version in classical terms reads as follows: Find the displacement field $u:[0,\infty)\times\Omega\mapsto\mathbb{R}^3$ and the non-symmetric plastic distortion tensor field $P : [0,\infty) \times \Omega \mapsto \mathbb{R}^{3\times 3}$, such that in $[0,\infty) \times \Omega$

$$\begin{aligned} -\operatorname{Div}\sigma &= f, & \Sigma^{\text{lin}} &= \Sigma^{\text{lin}}_{\text{e}} + \Sigma^{\text{lin}}_{\text{sh}} + \Sigma^{\text{lin}}_{\text{curl}}, \\ \sigma &= \Sigma^{\text{lin}}_{\text{e}}, & \Sigma^{\text{lin}}_{\text{e}} &= 2\mu \operatorname{sym}(\nabla u - P) + \lambda \operatorname{tr}(\nabla u - P)\,\text{id}, \\ \dot{P} &\in \partial\chi(\Sigma^{\text{lin}}), & \Sigma^{\text{lin}}_{\text{sh}} &= -2\mu \operatorname{sym} P, \quad \Sigma^{\text{lin}}_{\text{curl}} = -\mu L_c^2 \operatorname{Curl}\operatorname{Curl} P \end{aligned} \tag{1.1}$$

hold. The system is completed by the boundary conditions

$$u(t,x) = u_{\text{d}}(t,x), \quad \nu(x) \times P(t,x) = 0 \qquad \forall\,(t,x) \in [0,\infty) \times \partial\Omega$$

and the initial condition $P(0,x) = 0$ for all $x \in \Omega$.

Here, μ, λ are the elastic Lamé moduli and σ is the symmetric Cauchy stress tensor. Given body forces are denoted by f and displacement boundary conditions u_{d} on $\partial\Omega$ are supplied. The exterior normal to the boundary $\partial\Omega$ is denoted by ν and the plastic distortion P is required to satisfy row-wise the homogeneous tangential boundary condition which means that the boundary $\partial\Omega$ is a perfect conductor regarding the plastic distortion. Moreover, $\partial\chi$ is the subdifferential of the indicator function χ of the convex elastic domain with yield stress σ_{y}, i.e.,

$$\chi(\Sigma) = \begin{cases} 0 & , \text{ if } |\Sigma| \le \sigma_{\text{y}} \\ \infty & , \text{ otherwise} \end{cases}, \qquad \partial\chi(\Sigma) = \begin{cases} 0 & , \text{ if } |\Sigma| < \sigma_{\text{y}} \\ \mathbb{R}^+_0 \dfrac{\Sigma}{|\Sigma|} & , \text{ if } |\Sigma| = \sigma_{\text{y}} \\ \emptyset & , \text{ if } |\Sigma| > \sigma_{\text{y}} \end{cases}. \tag{1.2}$$

In general, $\Sigma^{\text{lin}}_{\text{curl}}$ is not symmetric even if P is symmetric. Thus, the plastic inhomogeneity is responsible for the plastic spin contribution in this rotationally invariant formulation. The mathematically suitable space for symmetric plastic distortion P is the classical space $\mathsf{H}(\text{curl};\Omega)$ for each row of P.

In the large scale limit $L_c \to 0$ we recover a classical elasto-plasticity model with local kinematic hardening and symmetric plastic strain $\varepsilon_P := \operatorname{sym} P$. Observe that the term $\Sigma^{\text{lin}}_{\text{curl}} = -\mu L_c^2 \operatorname{Curl}\operatorname{Curl} P$ acts as non-local kinematical backstress and constitutes a crystallographically motivated alternative to merely phenomenologically motivated backstress tensors. The term $-2\mu \operatorname{sym} P$ is a symmetric local kinematical backstress. Moreover, the driving stress for the plastic evolution Σ^{lin} is non-symmetric due to the presence of the second order gradients, while the local contribution σ, basically due to elastic lattice strains, remains symmetric.

Additionally, the infinitesimal local stress contributions are fully rotationally invariant (isotropic and objective) w.r.t. the transformation $(\nabla u, P) \mapsto (\nabla u + A(x), P + A(x))$ and the non-local stress contribution is still invariant w.r.t. the infinitesimal rigid transformation $(\nabla u, P) \mapsto (\nabla u + \bar{A}, P + \bar{A})$, where $\bar{A}, A(x) \in \mathfrak{so}(3)$.

Uniqueness of classical solutions for rate-independent and rate-dependent formulations of this model is shown in [20]. The more difficult existence question for the rate-independent model in terms of a weak reformulation is addressed in [21]. The related viscoplastic formulation of dislocation based gradient plasticity with kinematical hardening is treated in [26]. First numerical results for a simplified rate-independent irrotational formulation (no plastic spin, i.e., symmetric plastic distortion P) are presented in [25, 22]. For more on the basic invariance questions related to this model, see [41, 19]. In [7] the model has been extended to rate-independent isotropic hardening based on the concept of a dissipation function defined in terms of the equivalent plastic strain. From a modeling point of view, it is strongly preferable to again have only the symmetric (rate) part of the plastic distortion appear in the dissipation potential, see the discussion in [7].

The existence and uniqueness can be settled by recasting the model as a variational inequality, if it is possible to define a bilinear form which is coercive with respect to appropriate spaces. This program has been achieved for other variants of the model in [7]. It had to remain basically open for the above system (1.1). In this case, the appropriate space for the plastic distortion P is the completion

$$\overset{\circ}{\mathsf{H}}_{\mathrm{sym}}(\mathrm{Curl};\Omega)$$

of the linear space

$$\{P \in \mathsf{C}^\infty(\overline{\Omega},\mathbb{R}^{3\times 3})\ :\ \nu \times P|_{\partial\Omega} = 0\}$$

with respect to the norm $|\!|\!| \cdot |\!|\!|$, where

$$|\!|\!| P |\!|\!|^2 := \|\mathrm{sym}\, P\|^2_{\mathsf{L}^2(\Omega)} + \|\mathrm{Curl}\, P\|^2_{\mathsf{L}^2(\Omega)}\,. \tag{1.3}$$

Despite first appearance, this quadratic form indeed defines a norm as shown in [21]. Thus, $\overset{\circ}{\mathsf{H}}_{\mathrm{sym}}(\mathrm{Curl};\Omega)$ is a Hilbert-space and its elements have generalized row-wise vanishing tangential traces on $\partial\Omega$.

However, in this space it is not immediately obvious how to define a linear and bounded tangential trace operator. Since only $\|\mathrm{sym}\, P\|_{\mathsf{L}^2(\Omega)}$ appears, it is also not clear, how to control the skew-symmetric part of P. Therefore, the crucial embedding

$$\overset{\circ}{\mathsf{H}}_{\mathrm{sym}}(\mathrm{Curl};\Omega) \subset \mathsf{L}^2(\Omega)$$

is not clear as well. As a consequence of our main result of this paper we obtain that nevertheless

$$\overset{\circ}{\mathsf{H}}_{\mathrm{sym}}(\mathrm{Curl};\Omega) = \overset{\circ}{\mathsf{H}}(\mathrm{Curl};\Omega)$$

holds with equivalent norms in case the domain Ω is simply connected and has Lipschitz boundary.

For the proof of our main result (0.1) we combine techniques from electro-magnetic and elastic theory, namely Helmholtz' decomposition, Maxwell's compactness property (MCP) and Korn's inequality. Their basic variants are well known results which can be found in many books, e.g., in [14] and the literature cited there. More sophisticated and related versions are presented, e.g., in [11, 12, 13, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 44] for Maxwell's equations and [17] for Korn's inequality. Since these tools are crucial for our results we briefly look at their history.

As pointed out in the nice overview of W. Sprößig [40] H. von Helmholtz (1821-1894) founded a comprehensive development in the theory of projections methods mostly applied in, e.g., electro-magnetic or elastic theory or fluid dynamics. His famous theorem, called Helmholtz decomposition, states that any sufficiently smooth and sufficiently fast decaying vector field in $\mathbb{R}^3$ can be characterized by its rotation and divergence, i.e., it can be decomposed into a irrotational and a solenoidal part. A first uniqueness result was given by O. Blumenthal in [2]. Later, Hilbert and Banach space methods have been used to prove similar and refined decompositions of same type.

As to the MCP, in 1968 R. Leis [12] had considered the boundary value problem of total reflection for the inhomogeneous and anisotropic Maxwell system as well in bounded as in exterior domains. For bounded domains $\Omega \subset \mathbb{R}^3$ he was able to estimate all first derivatives of a vector field by the field itself, its divergence and its rotation in $\mathsf{L}^2(\Omega)$, i.e., there exists a constant $c > 0$, such that for all vector fields $v \in \mathsf{H}^1(\Omega)$

$$c \sum_{i,j=1}^{3} \|\partial_i v_j\|_{\mathsf{L}^2(\Omega)} \leq \|v\|_{\mathsf{L}^2(\Omega)} + \|\operatorname{curl} v\|_{\mathsf{L}^2(\Omega)} + \|\operatorname{div} v\|_{\mathsf{L}^2(\Omega)} \tag{1.4}$$

holds, provided that the boundary $\partial\Omega$ is sufficiently smooth. This estimate implies the MCP, namely

$$\begin{aligned} \mathsf{V}(\Omega) &:= \overset{\circ}{\mathsf{H}}(\operatorname{curl};\Omega) \cap \mathsf{H}(\operatorname{div};\Omega) \\ &= \{v \in \mathsf{L}^2(\Omega) \,:\, \operatorname{curl} v \in \mathsf{L}^2(\Omega),\, \operatorname{div} v \in \mathsf{L}^2(\Omega),\, \nu \times v|_{\partial\Omega} = 0\} \end{aligned} \tag{1.5}$$

is compactly embedded into $\mathsf{L}^2(\Omega)$ by Rellich's selection theorem, since $\mathsf{V}(\Omega)$ is a closed subspace of the Sobolev-Hilbert space $\mathsf{H}^1(\Omega)$. However, (1.4) becomes wrong if smoothness is not assumed. Leis encouraged some of his pupils [38, 43, 34, 46] to deal with electromagnetic problems, in particular with the MCP-question.

In 1969, H.D. Rinkens [38] (see also [14]) presented an example of a non-smooth two-dimensional domain, where the embedding of $\mathsf{V}(\Omega)$ into $\mathsf{H}^1(\Omega)$ is not possible. Another three-dimensional example had been found shortly later and is written down in a paper by J. Saranen [39].

Henceforth, proofs were looked for, which did not make use of an embedding of $\mathsf{V}(\Omega)$ into $\mathsf{H}^1(\Omega)$. In 1974, N. Weck [43] obtained a quite general result for 'cone-like' regions. Weck considered a generalization of Maxwells boundary value problem to Riemannian manifolds of arbitrary dimension N, going back to H. Weyl [45]. The cone-like regions have Lipschitz boundaries (but maybe not the other way round). However, polygonal boundaries are covered by Weck's result. In a joint paper by R. Picard, N. Weck and the third author [37], Weck's proof has been modified to even handle certain domains, which fail to be Lipschitz.

Proofs for Lipschitz domains have been given by M. Costabel [4] and C. Weber [42]. Costabel showed that $\mathsf{V}(\Omega)$ is continuously embedded into the fractional Sobolev space $\mathsf{H}^{1/2}(\Omega)$, which is compactly embedded into $\mathsf{L}^2(\Omega)$. Weber's proof has been modified by the third author [46] to obtain the MCP for domains with Hölder continuous boundaries (with exponent $q > 1/2$). Finally, there is a quite elegant result by R. Picard [34] who showed even in the generalized case that when the result holds for smooth boundaries it holds for Lipschitz boundaries as well.

2 Definitions and Preliminaries

Let Ω be a simply connected and bounded domain in $\mathbb{R}^3$ with Lipschitz continuous boundary $\Gamma := \partial\,\Omega$.

2.1 Functions and Vector Fields

The usual Lebesgue spaces of square integrable functions, vector or tensor fields on Ω with values in $\mathbb{R}$, $\mathbb{R}^3$ or $\mathbb{R}^{3\times 3}$, respectively, will be denoted by $\mathsf{L}^2(\Omega)$. Moreover, we introduce the standard Sobolev spaces

$$\begin{aligned}\mathsf{H}(\mathrm{grad};\Omega) &= \{u \in \mathsf{L}^2(\Omega) \,:\, \operatorname{grad} u \in \mathsf{L}^2(\Omega)\},\\ \mathsf{H}(\mathrm{curl};\Omega) &= \{v \in \mathsf{L}^2(\Omega) \,:\, \operatorname{curl} v \in \mathsf{L}^2(\Omega)\},\\ \mathsf{H}(\mathrm{div};\Omega) &= \{v \in \mathsf{L}^2(\Omega) \,:\, \operatorname{div} v \in \mathsf{L}^2(\Omega)\},\end{aligned}$$

where $\mathsf{H}(\mathrm{grad};\Omega)$ is often denoted by $\mathsf{H}^1(\Omega)$, and their closed subspaces

$$\mathring{\mathsf{H}}(\mathrm{grad};\Omega), \quad \mathring{\mathsf{H}}(\mathrm{curl};\Omega)$$

as completition under the respective graph norms of the scalar resp. vector valued space $\mathring{\mathsf{C}}^\infty(\Omega)$ of compactly supported and smooth test functions resp. vector fields. In the latter Sobolev spaces the usual homogeneous scalar resp. tangential boundary conditions

$$u|_\Gamma = 0, \quad \nu \times v|_\Gamma = 0$$

are generalized, where ν denotes the outer unit normal. Furthermore, we need the spaces of irrotational or solenoidal vector fields

$$\begin{aligned}\mathsf{H}(\mathrm{curl}_0;\Omega) &:= \{v \in \mathsf{H}(\mathrm{curl};\Omega) \,:\, \operatorname{curl} v = 0\},\\ \mathsf{H}(\mathrm{div}_0;\Omega) &:= \{v \in \mathsf{H}(\mathrm{div};\Omega) \,:\, \operatorname{div} v = 0\},\\ \mathring{\mathsf{H}}(\mathrm{curl}_0;\Omega) &:= \{v \in \mathring{\mathsf{H}}(\mathrm{curl};\Omega) \,:\, \operatorname{curl} v = 0\},\end{aligned}$$

where the index 0 indicates vanishing curl or div, respectively. All these spaces are Hilbert spaces. E.g., in classical terms we have in the weak sense

$$\mathring{\mathsf{H}}(\mathrm{curl}_0;\Omega) = \{v \in \mathsf{H}(\mathrm{curl};\Omega) \,:\, \operatorname{curl} v = 0,\, \nu \times v|_\Gamma = 0\}.$$

The most important tool for our analysis is the compact embedding

$$\mathring{\mathsf{H}}(\mathrm{curl};\Omega) \cap \mathsf{H}(\mathrm{div};\Omega) \hookrightarrow \mathsf{L}^2(\Omega), \tag{2.1}$$

which is often referred as 'Maxwell's compactness property'. As already mentioned in the introduction, there exists a rich amount of literature discussing Maxwell's compactness property, which holds even in the generalized case $\Omega \subset \mathbb{R}^N$ or for Riemannian manifolds

Ω using the calculus of differential forms. Among others we want to note the papers [43], [46], [34], [42], [37]. We also mention the nice overview in [14].

A first immediate consequence is that the space of so called 'harmonic Dirichlet fields'

$$\mathcal{H}(\Omega) := \overset{\circ}{\mathsf{H}}(\mathrm{curl}_0;\Omega) \cap \mathsf{H}(\mathrm{div}_0;\Omega) \tag{2.2}$$

is finite dimensional. A vector field v belonging to $\mathcal{H}(\Omega)$ means in classical terms that

$$\mathrm{curl}\, v = 0, \quad \mathrm{div}\, v = 0, \quad \nu \times v|_\Gamma = 0.$$

The dimension of $\mathcal{H}(\Omega)$ equals the second Betti number of Ω. Since we assume Ω to be simply connected, there are no Dirichlet fields besides zero.

By a usual indirect argument we achieve another immediate consequence:

Lemma 1 (Maxwell Estimate for Vector Fields) *There exists a positive constant c_m, such that for all $v \in \overset{\circ}{\mathsf{H}}(\mathrm{curl};\Omega) \cap \mathsf{H}(\mathrm{div};\Omega)$*

$$\|v\|_{\mathsf{L}^2(\Omega)} \le c_m \big(\|\mathrm{curl}\, v\|^2_{\mathsf{L}^2(\Omega)} + \|\mathrm{div}\, v\|^2_{\mathsf{L}^2(\Omega)} \big)^{1/2}.$$

By definition

$$\mathrm{grad}\, \overset{\circ}{\mathsf{H}}(\mathrm{grad};\Omega)^\perp = \mathsf{H}(\mathrm{div}_0;\Omega),$$

where $\perp$ denotes the orthogonal complement in $\mathsf{L}^2(\Omega)$. This implies

$$\overline{\mathrm{grad}\, \overset{\circ}{\mathsf{H}}(\mathrm{grad};\Omega)} = \mathsf{H}(\mathrm{div}_0;\Omega)^\perp,$$

where the closure is taken in $\mathsf{L}^2(\Omega)$. Hence, we obtain the Helmholtz decomposition

$$\mathsf{L}^2(\Omega) = \overline{\mathrm{grad}\, \overset{\circ}{\mathsf{H}}(\mathrm{grad};\Omega)} \oplus \mathsf{H}(\mathrm{div}_0;\Omega),$$

where $\oplus$ denotes the $\mathsf{L}^2(\Omega)$-orthogonal complement. The space $\mathrm{grad}\, \overset{\circ}{\mathsf{H}}(\mathrm{grad};\Omega)$ is already closed by Poincaré's estimate [14, p. 25], i.e.,

$$\exists\, c_p > 0 \quad \forall\, u \in \overset{\circ}{\mathsf{H}}(\mathrm{grad};\Omega) \quad \|u\|_{\mathsf{L}^2(\Omega)} \le c_p \|\mathrm{grad}\, u\|_{\mathsf{L}^2(\Omega)}, \tag{2.3}$$

which is implied by the compact embedding

$$\overset{\circ}{\mathsf{H}}(\mathrm{grad};\Omega) \hookrightarrow \mathsf{L}^2(\Omega), \tag{2.4}$$

i.e., Rellich's selection theorem, using an indirect argument. We have

Lemma 2 (Helmholtz Decomposition for Vector Fields) *The decomposition*

$$\mathsf{L}^2(\Omega) = \mathrm{grad}\, \overset{\circ}{\mathsf{H}}(\mathrm{grad};\Omega) \oplus \mathsf{H}(\mathrm{div}_0;\Omega)$$

holds.

2.2 Tensor Fields

We extend our calculus to 3×3-tensor (matrix) fields. For vector fields v with components in $\mathsf{H}(\mathrm{grad};\Omega)$ and tensor fields P with rows in $\mathsf{H}(\mathrm{curl};\Omega)$ resp. $\mathsf{H}(\mathrm{div};\Omega)$, i.e.,

$$v=\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix},\quad v_n\in\mathsf{H}(\mathrm{grad};\Omega),\quad P^T=[P_1\,P_2\,P_3],\quad P_n\in\mathsf{H}(\mathrm{curl};\Omega)\text{ resp. }\mathsf{H}(\mathrm{div};\Omega)$$

we define

$$\mathrm{Grad}\,v:=\begin{bmatrix}\mathrm{grad}^T v_1\\\mathrm{grad}^T v_2\\\mathrm{grad}^T v_3\end{bmatrix}=J_v,\quad \mathrm{Curl}\,P:=\begin{bmatrix}\mathrm{curl}^T P_1\\\mathrm{curl}^T P_2\\\mathrm{curl}^T P_3\end{bmatrix},\quad \mathrm{Div}\,P:=\begin{bmatrix}\mathrm{div}\,P_1\\\mathrm{div}\,P_2\\\mathrm{div}\,P_3\end{bmatrix},$$

where J_v denotes the Jacobian of v and ^{T} the transpose. We note that v and $\mathrm{Div}\,P$ are vector fields, whereas P, $\mathrm{Curl}\,P$ and $\mathrm{Grad}\,v$ are tensor fields. The corresponding Sobolev spaces will be denoted by

$$\mathsf{H}(\mathrm{Grad};\Omega),\qquad \overset{\circ}{\mathsf{H}}(\mathrm{Grad};\Omega),\qquad \mathsf{H}(\mathrm{Div};\Omega),\qquad \mathsf{H}(\mathrm{Div}_0;\Omega),$$
$$\mathsf{H}(\mathrm{Curl};\Omega),\qquad \overset{\circ}{\mathsf{H}}(\mathrm{Curl};\Omega),\qquad \mathsf{H}(\mathrm{Curl}_0;\Omega),\qquad \overset{\circ}{\mathsf{H}}(\mathrm{Curl}_0;\Omega).$$

Let us present our three crucial tools to prove the estimate. First we have obvious consequences from Lemmas 1 and 2:

Corollary 3 (Maxwell Estimate for Tensor Fields) *The estimate*

$$\|P\|_{\mathsf{L}^2(\Omega)}\leq c_m\big(\,\|\mathrm{Curl}\,P\|^2_{\mathsf{L}^2(\Omega)}+\|\mathrm{Div}\,P\|^2_{\mathsf{L}^2(\Omega)}\,\big)^{1/2}$$

holds for all tensor fields $P\in\overset{\circ}{\mathsf{H}}(\mathrm{Curl};\Omega)\cap\mathsf{H}(\mathrm{Div};\Omega)$.

Corollary 4 (Helmholtz Decomposition for Tensor Fields) *The decomposition*

$$\mathsf{L}^2(\Omega)=\mathrm{Grad}\,\overset{\circ}{\mathsf{H}}(\mathrm{Grad};\Omega)\oplus\mathsf{H}(\mathrm{Div}_0;\Omega)$$

holds.

The third important tool is Korn's first inequality. The simple variant which already meets our needs is the following:

Lemma 5 (Korn's First Inequality: $\overset{\circ}{\mathsf{H}}(\mathrm{Grad};\Omega)$-Variant) *For all* $v\in\overset{\circ}{\mathsf{H}}(\mathrm{Grad};\Omega)$

$$\|\mathrm{Grad}\,v\|_{\mathsf{L}^2(\Omega)}\leq\sqrt{2}\,\|\mathrm{sym}\,\mathrm{Grad}\,v\|_{\mathsf{L}^2(\Omega)}\,.$$

Here, we introduce the symmetric and skew-symmetric parts

$$\mathrm{sym}\,P:=\frac{1}{2}(P+P^T),\quad \mathrm{skew}\,P:=\frac{1}{2}(P-P^T)$$

of a tensor field $P=\mathrm{sym}\,P+\mathrm{skew}\,P$.

3 Main Results

For tensor fields $P \in \mathsf{H}(\mathrm{Curl};\Omega)$ we define the semi-norm

$$|||P||| := \left(\|\operatorname{sym} P\|^2_{\mathsf{L}^2(\Omega)} + \|\operatorname{Curl} P\|^2_{\mathsf{L}^2(\Omega)} \right)^{1/2}. \tag{3.1}$$

Lemma 6 *Let* $\hat{c} := \max\{2, \sqrt{5}c_m\}$. *Then, for all* $P \in \overset{\circ}{\mathsf{H}}(\mathrm{Curl};\Omega)$

$$\|P\|_{\mathsf{L}^2(\Omega)} \leq \hat{c}\, |||P||| .$$

Proof Let $P \in \overset{\circ}{\mathsf{H}}(\mathrm{Curl};\Omega)$. According to Corollary 4 we orthogonally decompose

$$P = \operatorname{Grad} v + Q \in \operatorname{Grad} \overset{\circ}{\mathsf{H}}(\mathrm{Grad};\Omega) \oplus \mathsf{H}(\mathrm{Div}_0;\Omega).$$

Then, $\operatorname{Curl} P = \operatorname{Curl} Q$ and we observe $Q \in \overset{\circ}{\mathsf{H}}(\mathrm{Curl};\Omega)$. By Corollary 3 we have

$$\|Q\|_{\mathsf{L}^2(\Omega)} \leq c_m \|\operatorname{Curl} P\|_{\mathsf{L}^2(\Omega)} . \tag{3.2}$$

Then, by Lemma 5 and (3.2) we obtain by orthogonality

$$\begin{aligned}
\|P\|^2_{\mathsf{L}^2(\Omega)} = \|\operatorname{Grad} v + Q\|^2_{\mathsf{L}^2(\Omega)} &= \|\operatorname{Grad} v\|^2_{\mathsf{L}^2(\Omega)} + \|Q\|^2_{\mathsf{L}^2(\Omega)} \\
&\leq 2\,\|\operatorname{sym}\operatorname{Grad} v\|^2_{\mathsf{L}^2(\Omega)} + \|Q\|^2_{\mathsf{L}^2(\Omega)} \\
&= 2\,\|\operatorname{sym}(P - Q)\|^2_{\mathsf{L}^2(\Omega)} + \|Q\|^2_{\mathsf{L}^2(\Omega)} \\
&\leq 4\,\|\operatorname{sym} P\|^2_{\mathsf{L}^2(\Omega)} + 5\,\|Q\|^2_{\mathsf{L}^2(\Omega)} \\
&\leq 4\,\|\operatorname{sym} P\|^2_{\mathsf{L}^2(\Omega)} + 5c_m^2\,\|\operatorname{Curl} P\|^2_{\mathsf{L}^2(\Omega)} ,
\end{aligned}$$

which completes the proof. □

The immediate consequence is

Theorem 7 *On* $\overset{\circ}{\mathsf{H}}(\mathrm{Curl};\Omega)$ *the norms* $\|\cdot\|_{\mathsf{H}(\mathrm{Curl};\Omega)}$ *and* $|||\cdot|||$ *are equivalent. In particular,* $|||\cdot|||$ *is a norm on* $\overset{\circ}{\mathsf{H}}(\mathrm{Curl};\Omega)$ *and there exists a positive constant c, such that*

$$c\,\|P\|^2_{\mathsf{H}(\mathrm{Curl};\Omega)} \leq |||P|||^2 = \|\operatorname{sym} P\|^2_{\mathsf{L}^2(\Omega)} + \|\operatorname{Curl} P\|^2_{\mathsf{L}^2(\Omega)}$$

holds for all $P \in \overset{\circ}{\mathsf{H}}(\mathrm{Curl};\Omega)$.

Setting $P := \operatorname{Grad} v$ we obtain

Remark 8 (Korn's First Inequality: Tangential-Variant) *For all* $v \in \overset{\circ}{\mathsf{H}}(\mathrm{Grad};\Omega)$

$$\|\operatorname{Grad} v\|_{\mathsf{L}^2(\Omega)} \leq \hat{c}\,\|\operatorname{sym}\operatorname{Grad} v\|_{\mathsf{L}^2(\Omega)} \tag{3.3}$$

holds by Lemma 6 since $\operatorname{Grad} \overset{\circ}{\mathsf{H}}(\mathrm{Grad};\Omega) \subset \overset{\circ}{\mathsf{H}}(\mathrm{Curl}_0;\Omega)$. *This is just Korn's first inequality from Lemma 5 with a larger constant* $\hat{c}$. *However,* (3.3) *already holds for all* $v \in \mathsf{H}(\mathrm{Grad};\Omega)$ *with* $\operatorname{Grad} v \in \overset{\circ}{\mathsf{H}}(\mathrm{Curl}_0;\Omega)$, *i.e., in classical terms* $\nu \times \operatorname{Grad} v|_\Gamma = 0$, *which then extends Lemma 5 through the weaker boundary condition.*

The elementary arguments above apply certainly to much more general situations. However, we will not explore this in the present paper.

4 The Two-Dimensional Case

Let Ω be a simply connected and bounded domain in $\mathbb{R}^2$ with Lipschitz continuous boundary. For tensor fields $P : \Omega \mapsto \mathbb{R}^{2\times 2}$ we define analogously the Curl-operator by

$$\operatorname{Curl} P = \operatorname{Curl} \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} = \begin{bmatrix} \operatorname{curl} [P_{11}\, P_{12}]^T \\ \operatorname{curl} [P_{21}\, P_{22}]^T \end{bmatrix} = \begin{bmatrix} \partial_1 P_{12} - \partial_2 P_{11} \\ \partial_1 P_{22} - \partial_2 P_{21} \end{bmatrix},$$

where now curl denotes the two dimensional scalar rotation and $\operatorname{Curl} P$ is a vector. With the appropriate changes, Lemma 6 and Theorem 7 hold as well. In particular, there exists a positive constant c, such that

$$c\,\|P\|_{\mathsf{L}^2(\Omega)} \leq \|\operatorname{sym} P\|_{\mathsf{L}^2(\Omega)} + \|\operatorname{Curl} P\|_{\mathsf{L}^2(\Omega)}$$

holds for all $P \in \overset{\circ}{\mathsf{H}}(\mathrm{Curl};\Omega)$.

During the preparation of our paper we got aware that a two-dimensional related result may be inferred from Garroni et al. [8]. Their result is also motivated by gradient plasticity models [3]. Instead of tangential boundary conditions on P they impose the normalization condition

$$\int_\Omega \operatorname{skew} P = 0. \tag{4.1}$$

Let us define the total variation measure of the distributional $\operatorname{Curl} P$ for $P \in \mathsf{L}^1(\Omega)$ by

$$|\operatorname{Curl} P|_\Omega := \sup_{\substack{v \in \overset{\circ}{\mathsf{C}}{}^1(\Omega) \\ |v|_{\mathsf{L}^\infty(\Omega)} \leq 1}} \langle P, \operatorname{CoGrad} v \rangle_{\mathsf{L}^2(\Omega)},$$

where

$$\operatorname{cograd} u := R \operatorname{grad} u = \begin{bmatrix} \partial_2 u \\ -\partial_1 u \end{bmatrix}, \quad R := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$
$$\operatorname{CoGrad} v := \begin{bmatrix} \operatorname{cograd}^T v_1 \\ \operatorname{cograd}^T v_2 \end{bmatrix} = \begin{bmatrix} \partial_2 v_1 & -\partial_1 v_1 \\ \partial_2 v_2 & -\partial_1 v_2 \end{bmatrix}.$$

We note

$$\langle P, \operatorname{CoGrad} v \rangle_{\mathsf{L}^2(\Omega)} = \int_\Omega P_{11}\,\partial_2 v_1 - P_{12}\,\partial_1 v_1 + P_{21}\,\partial_2 v_2 - P_{22}\,\partial_1 v_2.$$

Using partial integration, i.e., $\langle P, \operatorname{CoGrad} v \rangle_{\mathsf{L}^2(\Omega)} = \langle \operatorname{Curl} P, v \rangle_{\mathsf{L}^2(\Omega)}$, it is easy to see that $|\operatorname{Curl} P|_\Omega = \|\operatorname{Curl} P\|_{\mathsf{L}^1(\Omega)}$ if $\operatorname{Curl} P \in \mathsf{L}^1(\Omega)$. In [8, Th. 9] they show that for Ω having a Lipschitz boundary and a special 'slicing' property, there exists a constant $c > 0$, such that

$$c\,\|P\|_{\mathsf{L}^2(\Omega)} \leq \|\operatorname{sym} P\|_{\mathsf{L}^2(\Omega)} + |\operatorname{Curl} P|_\Omega$$

holds for all $P \in \mathsf{L}^1(\Omega)$ with (4.1). Their proof uses essentially the fact that in $\mathbb{R}^2$ the operators curl and div can be exchanged by a simple transformation, i.e., the identity $\operatorname{curl} v = -\operatorname{div} Rv$ holds for vector fields v. Thus, such a strong result may not be true in higher space dimensions $N \geq 3$ and it is open whether the normalization condition (4.1) can be exchanged with tangential boundary conditions.

5 Further Applications: Micromorphic Model

Let us define

$$\overset{\circ}{\mathsf{H}}_{\mathrm{sym}}(\mathrm{Curl};\Omega) := \overline{\{P \in \mathsf{C}^\infty(\overline{\Omega}, \mathbb{R}^{3\times 3}) \,:\, \nu \times P|_\Gamma = 0\}},$$

where the closure is taken in the norm $|||\cdot|||$. Since

$$\overset{\circ}{\mathsf{C}}{}^\infty(\Omega) \subset \{P \in \mathsf{C}^\infty(\overline{\Omega}, \mathbb{R}^{3\times 3}) \,:\, \nu \times P|_\Gamma = 0\}$$

it follows from the last section that

$$\overset{\circ}{\mathsf{H}}_{\mathrm{sym}}(\mathrm{Curl};\Omega) = \overset{\circ}{\mathsf{H}}(\mathrm{Curl};\Omega) \tag{5.1}$$

with equivalent norms.

In the theory of extended continuum mechanics we encounter the micromorphic approach. A subvariant of this model can be written in the form of a minimization problem for two fields, i.e., the classical displacement $u : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$ and the micromorphic tensor field $P : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^{3\times 3}$. The additional field may be needed in the description of foams and bones [10, 23, 18, 24, 16]. The problem is to find the pair (u, P) such that

$$\int_\Omega \mu|\operatorname{sym}(\nabla u - P)|^2 + \frac{\lambda}{2}|\operatorname{tr}(\nabla u - P)|^2 - f\cdot u$$
$$+ h^+\big(\mu|\operatorname{sym} P|^2 + \frac{\lambda}{2}|\operatorname{tr} P|^2\big) + \mu L_c^2|\operatorname{Curl} P|^2 \longrightarrow \min$$

subject to the boundary conditions of place $u|_\Gamma = 0$ and $\nu \times P|_\Gamma = 0$. Here, the problem is driven by the body force f and $L_c > 0$ has dimensions of length, $h^+ > 0$ is a non-dimensional factor and μ, λ are the Lamé-constants of the material. With our theory at hand one can show that the unique solution satisfies $u \in \overset{\circ}{\mathsf{H}}(\mathrm{Grad};\Omega)$ and $P \in \overset{\circ}{\mathsf{H}}(\mathrm{Curl};\Omega)$. In order that the model is invariant with respect to superposed infinitesimal rigid rotations, i.e., $(\nabla u, P) \mapsto (\nabla u + A, P + A)$ for constant $A \in \mathfrak{so}(3)$, the symmetric local contribution $\operatorname{sym} P$ is mandatory and leads us to consider $\overset{\circ}{\mathsf{H}}_{\mathrm{sym}}(\mathrm{Curl};\Omega)$ in the first place. But fortunately (5.1) holds. Provided that $h^+ = 1$, this formulation is a relaxed formulation of linear elasticity, since the stored energy will always be less than the stored energy for the corresponding linear elastic formulation: Just take $P = \nabla u$, where u is the classical solution. This remains true even in the formal limit $L_c \to \infty$.

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